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SOME PROBLEMS IN NONLINEAR ANALYSIS

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The work accomplished under the grant by M. Crandall is reported on in the published papers listed below. We briefly summarize the nature of the work, referring to the published papers.

Papers [4] and [6] are a continuation of the authors' program to provide a theoretical basis for Hamilton-Jacobi equations in infinite dimensions, with an eye to developing a theoretical foundation for dynamic programming in infinite dimensional control and differential games. These works were the first to establish existence, uniqueness, etc., under conditions allowing the dynamics to include partial differential equations of evolution. In [4] the theory relied on weak continuity of the solutions in order to produce the extrema needed for the theory; this approach allowed one to extend results previously obtained for equations without terms involving unbounded operators. These earlier results employed results of Ekeland and Stegall on perturbed optimization. However, this restricted the theory in such a way that it did not apply, for example, to dynamics which were pde's in all of $\mathbb{R}^n$. In [6] this restriction was overcome by introducing the notion of "B"-continuity, where B is an operator occurring in the theory and used to parameterize the continuity inherited by the value function from the dynamics. Existence and uniqueness were shown again using perturbed optimization. Other workers are now entering this rich arena; we mention in particular D. Tataru who was stimulated by this series to make some dramatic recent contributions.

The paper [5] perfects a result of [2] which allowed a significant simplification of the presentation of the theory of viscosity solutions of fully nonlinear second order partial differential equations. This theory is now an enormously active area, finding new applications in asymptotics, geometry, control, etc., almost daily. The theory allows functions which are not necessarily differentiable to be solutions of fully nonlinear partial differential equations for which one formally has a maximum principle for smooth solutions. Functions of this sort arise as the value functions in control, stochastic control, and differential games as well as in geometry. The key analytical fact which "makes the theory go" lies rather deep and was recognized relatively late in the game to correspond to a property of semicontinuous functions. This property may be stated without reference to the equations in the background, just as the fact the matrix of second derivatives has a sign at an extremum does not require partial differential equations to formulate. This property of semicontinuous functions is isolated and proved in [5]. Using the results as formulated in [5], [9] is a primarily (but not completely) expository presentation of the theory. Indeed, we expect [9] to have a very substantial impact as it simplifies many aspects of working with viscosity solutions and presents the main ideas and results in a congenial and organized way for the first time. Preliminary response indicates that it will be widely read. It will appear in the Bulletin of the American Mathematical Society as a "research expository article". The paper [8] is a related short proceedings article intended to advertise these developments to its audience. Papers [1] and [3] are in the same arena, but are concerned with issues of growth of solutions of fully nonlinear equations in $\mathbb{R}^n$.

The paper [7], although a proceedings paper, is largely original. It provides an abstract functional analytic framework for the discussion of evolutions generated by operators like the Laplacian or, more generally, the p-Laplacian, which behave well in almost all spaces.

In the process, a simple outline of some interpolation theory is given, with applications showing that classical results for linear mappings may be obtained in a simple way even in the nonlinear case and new generation and perturbation theorems are obtained.

PUBLICATIONS LIST - M. G. Crandall

[1] M. G. Crandall, R. Newcomb and Y. Tomita, Existence and uniqueness of viscosity solutions of degenerate quasilinear elliptic equations in $\mathbb{R}^n$ *Applicable Analysis* 34 (1989), pp. 1-23.

[2] M. G. Crandall, Quadratic forms, semidifferentials and viscosity solutions of fully nonlinear elliptic equations, *Ann. I.H. P. Anal. Non. Lin.* 6 (1989), pp. 419-435.

[3] M. G. Crandall and P. L. Lions, Quadratic growth of solutions of fully nonlinear second order equations on $\mathbb{R}^n$, *Diff. Int. Equa.* 3 (1990), pp. 601-616.

[4] M. G. Crandall and P. L. Lions, Hamilton-Jacobi equations in infinite dimensions: Part IV. Unbounded linear terms, *J. Func. Anal.* 90 (1990), pp. 237–283.

[5] M. G. Crandall and H. Ishii, The maximum principle for semicontinuous functions, *Diff. and Int. Equations* 3 (1990), pp. 1001-1014.

[6] M. G. Crandall and P. L. Lions, Hamilton-Jacobi equations in infinite dimensions: Part V. B-continuous solutions, *J. Func. Anal.*, in press.

[7] Ph. Benilan and M. G. Crandall, Completely Accretive Operators, Proceedings of the International Symposium on Semigroup Theory and Partial Differential Equations, Delft, Netherlands, 1989, Marcel Dekker, in press.

[8] M. G. Crandall, The Maximum principle, semicontinuity and nonlinear pde's, Proceedings of the 29th IEEE Conference on Decision and Control, Honolulu, 1990.

[9] M. G. Crandall, H. Ishii and P. L. Lions, Users guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc.*, to appear.

P. H. Rabinowitz

Our research has involved developing novel methods in the calculus of variations and applying them to problems in dynamical systems and partial differential equations. A large number of problems has been studied. In [1-4,6-9,11-12], various questions involving periodic solutions of Hamiltonian systems are treated. E.g. [1] concerns a priori bounds for the period in terms of the action integral; [2] obtains multiplicity results for solutions when the corresponding functional has a $\mathbb{Z}^n$ symmetry; [4], [6], and [8] concern the existence of solutions of prescribed energy; [3], [7], [9], [11-12] deal with the existence of solutions of prescribed period for various kinds of singular Hamiltonian systems such as arise in celestial mechanics. Papers [8], [10], [13-17] study the existence of various kinds of connecting orbits for Hamiltonian systems such as heteroclinic and homoclinic orbits. Some of the techniques involved in [17] are extended to get the existence of infinitely many multi-bump solutions (or solutions of homoclinic type) for a family of semilinear elliptic problems in [18].

Paper [5] is a survey of various results for superlinear problems. Finally in nearly completed research [19], ideas from [13] and [18] are used to get standing wave solutions of nonlinear Schrödinger equations.

PUBLICATIONS LIST for P. H. Rabinowitz:

1. A priori bounds for periodic solutions of a class of Hamiltonian systems, (with V. Benci), Ergodic Theory Dynamical Systems **8** (1988), 27–31.
2. On a class of functionals invariant under a Z^n action, Trans. Amer. Math. Soc. **310** (1988), 303–311.
3. A minimax method for a class of Hamiltonian systems with singular potentials, (with A. Bahri), J. Funct. Anal. **82** (1989), 412–428.
4. Periodic solutions of prescribed energy of Hamiltonian systems PDE, Springer Lecture Notes No. 1324 (1988), 253–262.
5. A survey of some superlinear problems, Nonlinear Diffusion Eq. and Their Equilibrium States, W. M. Ni, ed., Springer Verlag (1988), 217–234.
6. The prescribed energy problem for periodic solutions of Hamiltonian systems, Contemp. Math. **81** (1988), 183-191.
7. Periodic solutions of Hamiltonian systems of 3-body type, (with A. Bahri), (110 pages), Analyse Nonlinéaire, to appear.
8. Periodic and heteroclinic orbits for a periodic Hamiltonian system, Analyse Nonlinéaire **6** (1989), 331–346.
9. Orbits periodiques des systemes Hamiltonians singuliers des type de celui des trois corps, C. R. Acad. Sci. Paris **310** ser. I, (1990), 155–160.
10. Homoclinic orbits for a class of Hamiltonian systems, Proc. Roy. Soc. Edinburgh **114A** (1990), 33–38.
11. Periodic solutions of some problems of 3-body type, (with A. Bahri), Proc. Workshop on Variational Problems, Paris 1988, to appear.
12. Periodic solutions for some forced Hamiltonian systems, in Analysis, et cetera (P. H. Rabinowitz & E. Zehnder, editors), Academic Press, 1990, 521–544.
13. Some results on connecting orbits for a class of Hamiltonian systems, (with K. Tanaka), Math. Z., to appear.
14. A variational approach to heteroclinic orbits for a class of Hamiltonian systems, in Festschrift in honor of J.-L. Lions 60th birthday, to appear.
15. Some recent results on heteroclinic and other connecting orbits of Hamiltonian systems, Proc. Int. Conf. on Hamiltonian Systems and Nonlinear Elliptic Partial Differential Equations, to appear.
16. Variational approaches to the existence of homoclinic solutions, Festschrift in honor of S. Smale's 60th birthday, to appear.
17. Homoclinic orbits for 2nd order Hamiltonian systems possessing superquadratic potentials, (with V. Coti-Zelati), submitted.
18. Homoclinic type solutions for a semilinear elliptic PDE on $\mathbb{R}^n$, (with V. Coti-Zelati), submitted.
19. On a class of nonlinear Schrödinger equations, in progress.